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CONTINUANT EXPRESSIONS FOR $\sqrt{a^2 + b}$ AND $(\sqrt{a^2 + b} + a)^n$.

By L. H. RICE.

The line of reasoning employed in a paper by Muir* is applicable to a more general expression than that with which his paper was concerned.

We shall first show that *if the positive integral powers of $\sqrt{a^2 + b} + a$ be taken, and the expansion of each be separated into two parts, rational and irrational, then the ratio of the rational portion to the coefficient of $\sqrt{a^2 + b}$ approaches as a limit $\sqrt{a^2 + b}$ or $-\sqrt{a^2 + b}$, as the index of the power approaches infinity, according as a is positive or negative.*

Manifestly,

$$(\sqrt{a^2 + b} + a)^n = \frac{(\sqrt{a^2 + b} + a)^n + (-1)^{n-1}(\sqrt{a^2 + b} - a)^n}{2} + \frac{(\sqrt{a^2 + b} + a)^n + (-1)^n(\sqrt{a^2 + b} - a)^n}{2};$$

and in this expression the first fraction always contains $\sqrt{a^2 + b}$ as a factor, while the second fraction is always rational. Consequently we write

$$(\sqrt{a^2 + b} + a)^n = \frac{(\sqrt{a^2 + b} + a)^n + (-1)^{n-1}(\sqrt{a^2 + b} - a)^n}{2\sqrt{a^2 + b}} \sqrt{a^2 + b} + \frac{(\sqrt{a^2 + b} + a)^n + (-1)^n(\sqrt{a^2 + b} - a)^n}{2}, \quad (1)$$

thereby separating the expansion as specified. The n th convergent,

$$\frac{(\sqrt{a^2 + b} + a)^n + (-1)^n(\sqrt{a^2 + b} - a)^n}{(\sqrt{a^2 + b} + a)^n + (-1)^{n-1}(\sqrt{a^2 + b} - a)^n} \sqrt{a^2 + b},$$

may be put into either of the forms

* Muir, Thos., "Note on a theorem regarding a series of convergents to the roots of a number," Proc. Roy. Soc. Edin., vol. XIX, p. 15.

$$\frac{1 + (-1)^n \left(\frac{\sqrt{a^2 + b} - a}{\sqrt{a^2 + b} + a} \right)^n}{1 + (-1)^{n-1} \left(\frac{\sqrt{a^2 + b} - a}{\sqrt{a^2 + b} + a} \right)^n} \sqrt{a^2 + b}, \quad \frac{\left(\frac{\sqrt{a^2 + b} + a}{\sqrt{a^2 + b} - a} \right)^n + (-1)^n}{\left(\frac{\sqrt{a^2 + b} + a}{\sqrt{a^2 + b} - a} \right)^n + (-1)^{n-1}} \sqrt{a^2 + b}.$$

If a is positive, the first form shows that the limit is $\sqrt{a^2 + b}$; if a is negative, the second form shows that the limit is $-\sqrt{a^2 + b}$.

Ramus,* in 1856, obtained a result which we may express in the form

$$\begin{vmatrix} a & b & & \\ -1 & a & b & \\ & -1 & a & b \\ & & \ddots & \ddots \end{vmatrix}_{n-1} = \frac{1}{\sqrt{a^2 + 4b}} \left[\left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^n - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \right],$$

or, replacing a by $2a$, and making a further obvious modification,

$$\begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \ddots & \ddots \end{vmatrix}_{n-1} = \frac{(\sqrt{a^2 + b} + a)^n + (-1)^{n-1} (\sqrt{a^2 + b} - a)^n}{2\sqrt{a^2 + b}}.$$

We also have, from the properties of continuants,

$$\begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \ddots & \ddots \end{vmatrix}_n = a \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \ddots & \ddots & \ddots \end{vmatrix}_{n-1} + b \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \ddots & \ddots & \ddots \end{vmatrix}_{n-2},$$

whence, since $b = (\sqrt{a^2 + b} + a)(\sqrt{a^2 + b} - a)$, the continuant on the left of the last equation is equal to

$$\frac{a(\sqrt{a^2 + b} + a)^n + (-1)^{n-1} a(\sqrt{a^2 + b} - a)^n + (\sqrt{a^2 + b} - a)(\sqrt{a^2 + b} + a)^n + (-1)^{n-2} (\sqrt{a^2 + b} + a)(\sqrt{a^2 + b} - a)^n}{2\sqrt{a^2 + b}} = \frac{(\sqrt{a^2 + b} + a)^n + (-1)^n (\sqrt{a^2 + b} - a)^n}{2},$$

* Ramus, C., "Determinanternes Anvendelse til at bestemme hoven for de convergerende Brøker." Oversigt . . . danske Vidensk. Selsk. Forhandl. . . Kjøbenhavn, pp. 106-119. Muir's Theory of Determinants, vol. II, p. 427.

which is the rational term in (1). We may therefore rewrite (1) in the forms

$$\begin{aligned}
 (\sqrt{a^2+b}+a)^n &= \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \ddots & \ddots & \ddots \end{vmatrix}_{n-1} \sqrt{a^2+b} + \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \ddots & \ddots \end{vmatrix}_n \\
 &= \begin{vmatrix} 1 & \sqrt{a^2+b} & & \\ -1 & a & b & \\ & -1 & 2a & b \\ & & -1 & 2a & b & \\ & & & \ddots & \ddots & \ddots \end{vmatrix}_{n+1}. \quad (A)
 \end{aligned}$$

We found that

$$\sqrt{a^2+b} = \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \ddots & \ddots \end{vmatrix} \div \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \ddots & \ddots & \ddots \end{vmatrix} \quad (a > 0),$$

and

$$\sqrt{a^2+b} = - \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \ddots & \ddots \end{vmatrix} \div \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \ddots & \ddots & \ddots \end{vmatrix} \quad (a < 0).$$

By the rule for changing the signs of the principal diagonal elements of a continuant, the latter equation becomes

$$\sqrt{a^2+b} = \begin{vmatrix} |a| & b & & \\ -1 & 2|a| & b & \\ & -1 & 2|a| & b \\ & & \ddots & \ddots \end{vmatrix} \div \begin{vmatrix} 2|a| & b & & \\ -1 & 2|a| & b & \\ & \ddots & \ddots & \ddots \end{vmatrix},$$

which holds for both positive and negative values of a . Hence we have, finally,

$$\sqrt{a^2+b} = |a| + \frac{b}{2|a|} + \frac{b}{2|a|} + \dots \quad (B)$$

A part of this result will be seen to furnish a proof of the truth of an equation put forth as a problem in Chrystal's Algebra, Part II, Exs. XXXI, No. (9).

In the proof leading up to equation (B) it is a necessary condition, in case b is negative, that $a^2 > |b|$.

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